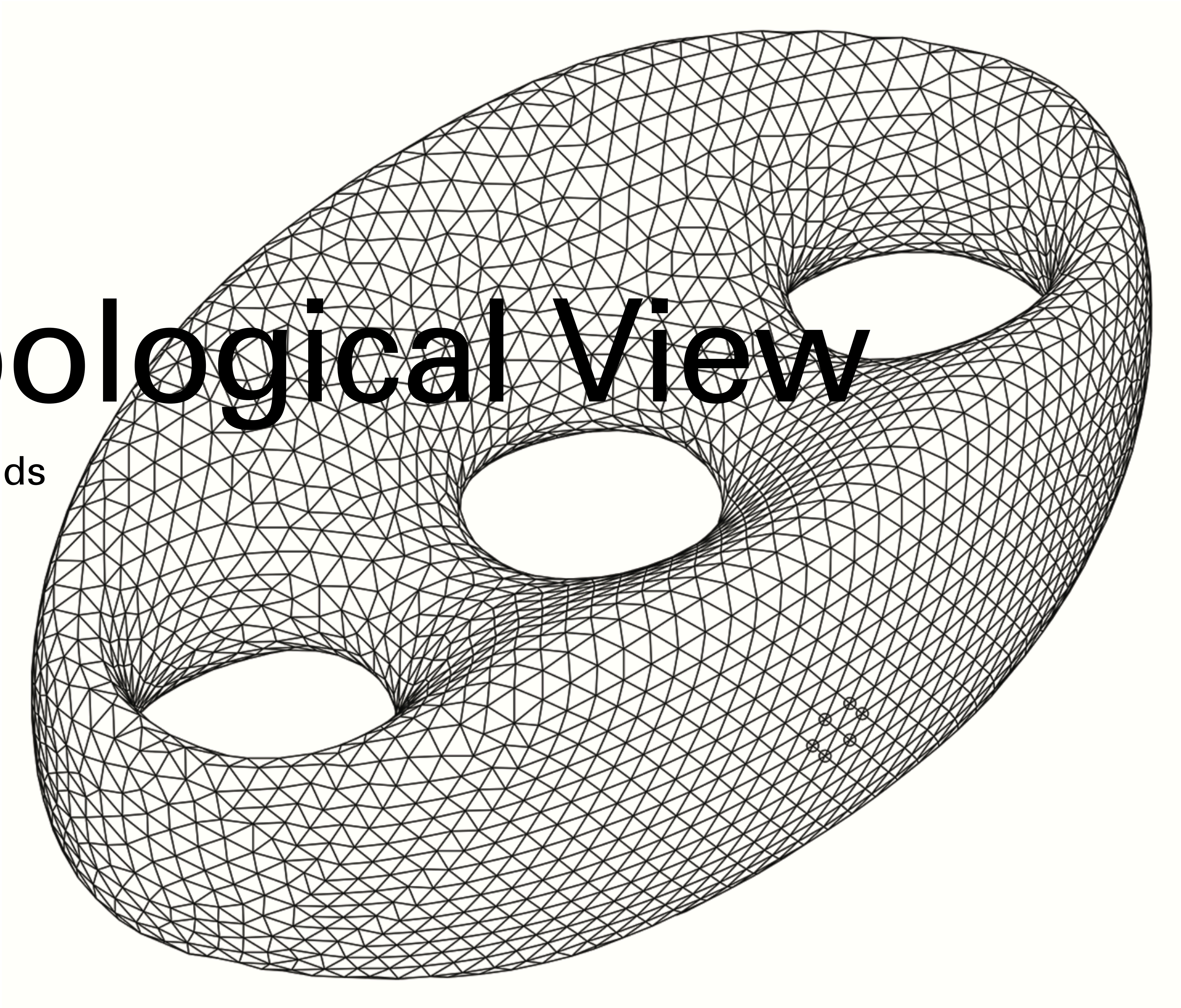
The Topological View

On the Realizability of Worlds



AA Cavia, Antikythera Program. 09.02.23.

Research Questions

Q: What are the epistemological limits of AI?

Entails answers to some of the following:

What does it mean to say an agent ‘understands’ language?
What does it mean to say that a concept has ‘content’?
What is representation and what role does it play in cognition?
How is language anchored in the world around us?

Well trodden ground in academic philosophy

Which languages are computational in nature?
What kind of mechanisms are computational and why?
What is the relation between computation, logic, and mathematics?
What semantics is adequate for describing computational states?

Left to technical disciplines (computational linguistics, CS)

What is Computation?

Canonical models reveal three facets, by turns:

- Mathematical, General Recursive Functions (Gödel)
- Linguistic, Lambda Calculus (Church)
- Mechanistic, Universal Machine (Turing)

There is cause to be dissatisfied by taking the equivalence of these models as an adequate ‘definition’ of computation. To gain conceptual traction on its inferential limits we need a holistic viewpoint. Rather than seeking further definitions we need to construct a worldview.

By ‘world’ I am alluding to a local “site for the identification of beings” (Badiou). A world in this sense is a logical space (a topos) which presupposes an identity function and one or more entities. It is a construction which ‘makes sense’ of our registrations under the “tribunal the experience” (McDowell).

What is Computation?

1. Consider it a diagnosis of contingency, generating distinct notions of:

- Incompleteness, μ operator (Gödel)
- Inconsistency, Type Theory (Church)
- Undecidability, Effective computability (Turing)

2. Consider it a mode of explanation or form of inference—distinct from classical mathematics or causal explanation—which we can call computational reason.

An inferential theory of computation would need to integrate the two views. The properties of this theory would include an adequate semantics, an account of knowledge formation, a rendering of its relation to logic, and some conceptual resources for unifying statistical inference with symbolic approaches.

Proposal: Computational reason not only defines the “space of reasons” (Sellars) which are computational in nature, but represents the very genesis of spaces as topological sites from which truths (invariance) can be realized.

The Realizability of Worlds

AND	0	1
0	0	0
1	0	1

OR	0	1
0	0	1
1	1	1

Tarski and the metalinguistic approach to the semantics of formal languages (ie. model theory) is centered on truth values. Computational states (eg. voltages on silicon) are interpreted by Boolean truth tables which are extrinsic to the axiomatic system.

Kleene, Heyting, Brouwer, Kolmogorov and the proof theoretic tradition finds its computational expression in the notion of realizability. It replaces truth with a many valued semantic theory centered on operations or procedures.

This richer semantics allows us to consider the meaning of computational states without reducing them to binary truth values, in the process opening up computation to a much more expressive inferential landscape.

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Computation is the coming together of logic and matter, of logiciel and matériel—it sits at the nexus of logical and material laws.

The realizability interpretation of logic insists that the meaning of a logical proposition is synonymous with its methods of construction. First introduced by Kleene (1945) as a means of interpreting statements in intuitionistic number theory.

Under the realizability interpretation, logical propositions are said to be ‘realized’ by their corresponding types:

$$(\mathrm{M}, \mathrm{N}) \Vdash A \,\&\, B$$

Types (M, N) in turn represent all the proofs or programs which can show that proposition to be true (Martin-Löf).

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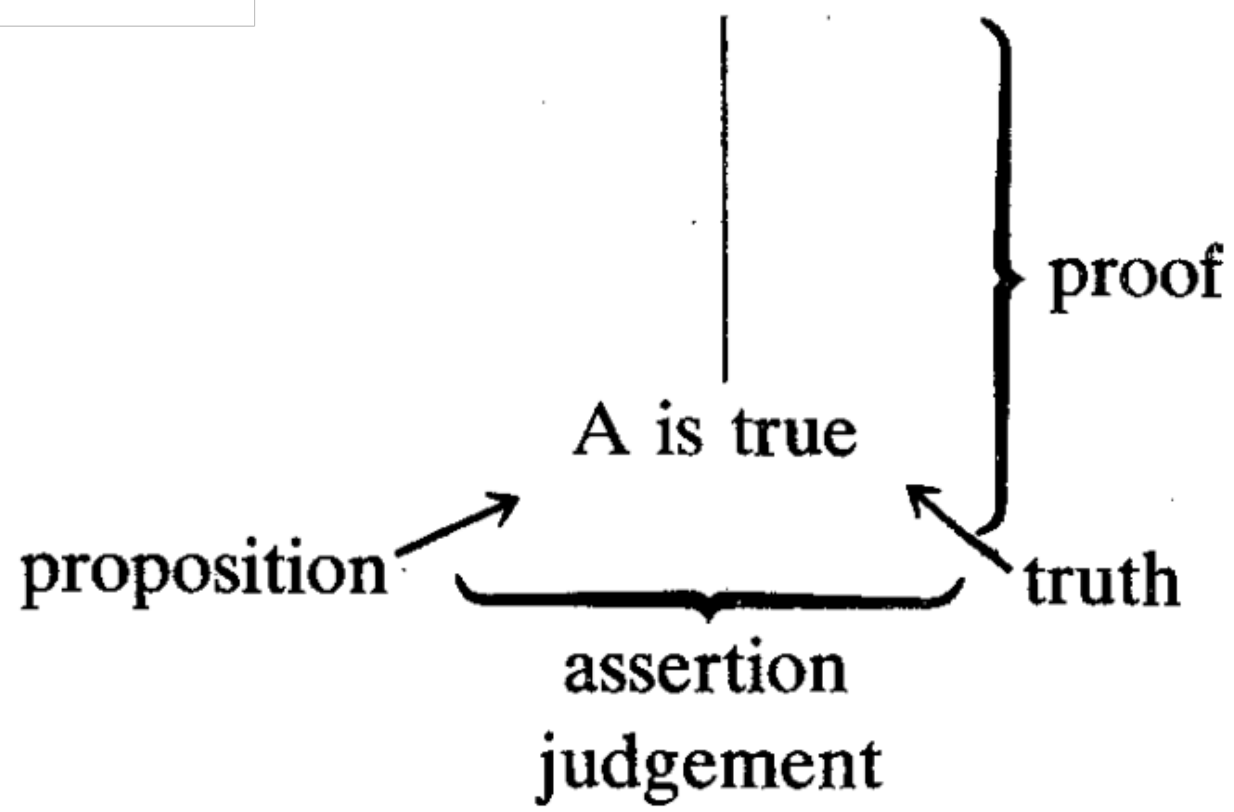

The realizability operator introduces a computationalist perspective on judgement formation and imbues types with inferential properties. Types are said to be 'inhabited' by the set of operations or procedures that act as 'witnesses' to the truth of a proposition (Martin-Löf).

Type theory emerges as a foundational system with its own constructive semantics. The 'assignment' of a term to a type is synonymous with providing a justification for a proposition, furnishing us with a theory of conceptual content: the 'content' of a proposition is all the ways we have of justifying it (all the programs that output instances of the corresponding type).

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The Space of Reasons

For Sellars, empirical descriptive vocabulary is parasitic on modal claims which in turn rest on normative commitments, which is to say, speech acts are embedded within notions of how the world could be and how it ought to be.

The ‘givenness’ of the contents of experience cannot be decoupled from our conceptual abilities as discursive language bearing agents. We must be vigilant of the “myth of the given” in all its forms, experience is not ‘given’ in the first instance and our modes of knowledge acquisition are peculiarly human in this respect.

As such, Sellars is broadly a nominalist on universals, but details how they can be ‘constructed’ from particulars. We can say that the projectibility of a theory defines its capacity to act globally as a general law in a given space, and that propositions are always locally embedded (the locality of truth).

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The Space of Reasons

To navigate the space of reasons is to deploy provisional inferential strategies which are local, time bound, and geometric. The static figure of the axiomatic is supplanted by a dynamic logic we can call inferential.

“In an axiomatic system, a list of axioms is provided… on the basis of which to deduce theorems. Axioms are judgments furnishing premises for inferences. In a natural deduction system (Gentzen) one is provided not with axioms but instead with a variety of rules of inference governing the sorts of inferential moves from premises to conclusions that are legitimate in the system. In natural deduction, one must furnish the premises oneself; the rules only tell you how to go on.”

- Macbeth, D., 2014. Realizing Reason: A Narrative of Truth and Knowing. OUP Oxford. p. 73

The Space of Reasons

A space is no longer a naive Euclidean notion assumed in advance of our axioms, but rather a “space of models” that satisfy a given geometric theory (Vickers). The topological view is an assertion that all space comes with an attendant structure, moreover that the structure is primary (locale theory).

A major shift in our mathematized conception of space comes with the non-euclidean spaces of Riemannian manifolds. Another rupture is occassioned by Hilbert’s breakthrough in showing how calculus and geometry can be performed in high-dimensional spaces. A third critical development is expressed by Grothendieck’s notion of a ‘topos’.

Topology is in a sense "the liberation of geometry, ‘freed’ at last from the physical universe” (Châtelet) and the role of computation is now cast as the navigation of topological sites.

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The Topological Turn

“In the usual conception of geometry a space is a set of points equipped with extra structure, such as metric or topology. But we can switch to a different view in which the extra structure is primary and points are derived ideal objects. For example, a topological space is not viewed as a set of points with a topology anymore, but rather just the topology, given as an abstract lattice with suitable properties, known as a locale. In constructive mathematics such treatment of the notion of space is much preferred to the usual one.”

- Bauer, A., 2013. Intuitionistic Mathematics and Realizability in the Physical World. In: A Computable Universe: Understanding and Exploring Nature as Computation. pp. 143-157.

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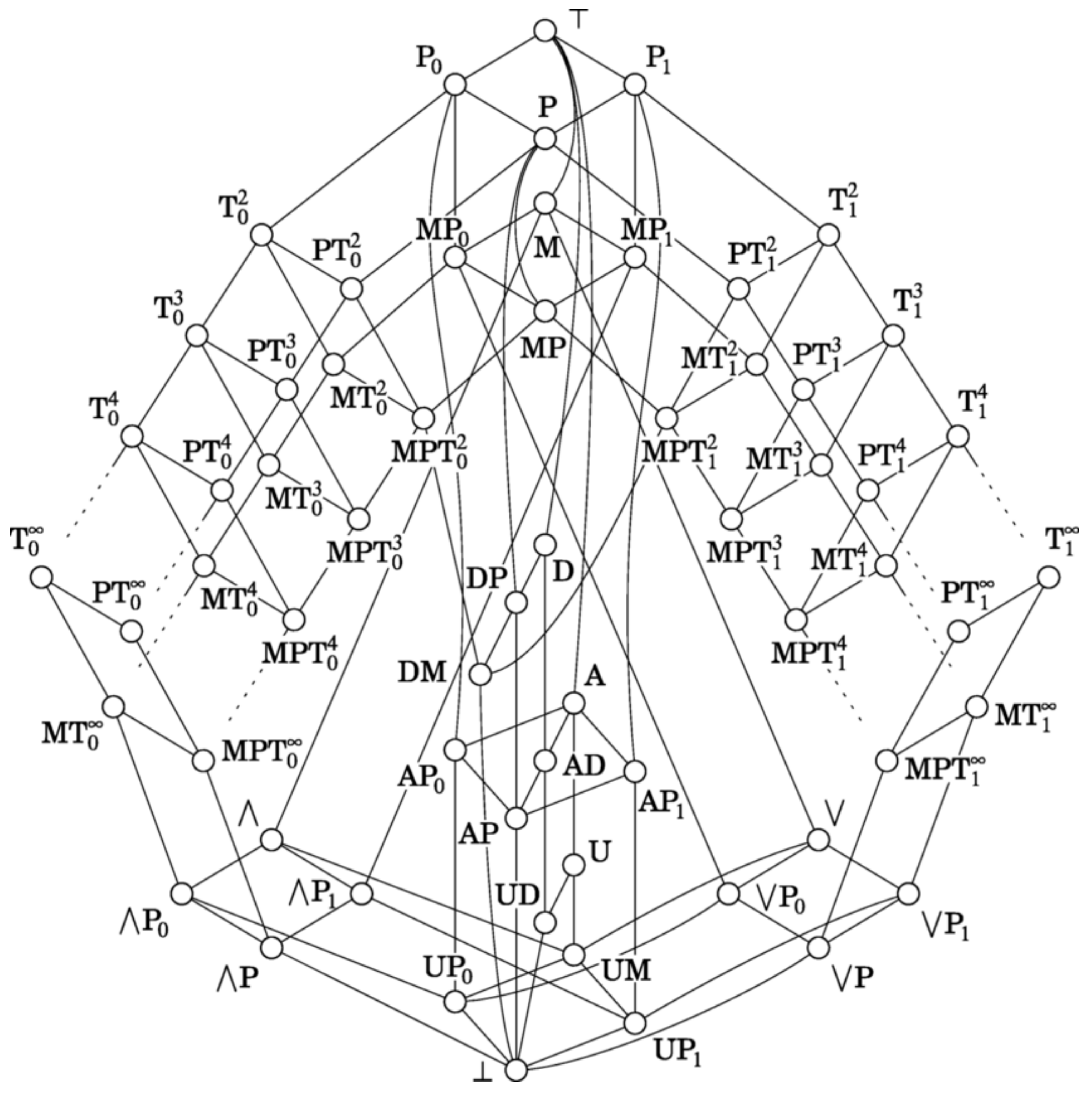

The above is a Hasse diagram of Post's Lattice of Boolean clones. Each node represents a Boolean algebra which is said to enter into 'covering' relations with those below it. In locale theory, a special kind of lattice known as a frame is put into correspondence with a topological space, creating an object known as a 'locale' which is both geometric and algebraic.

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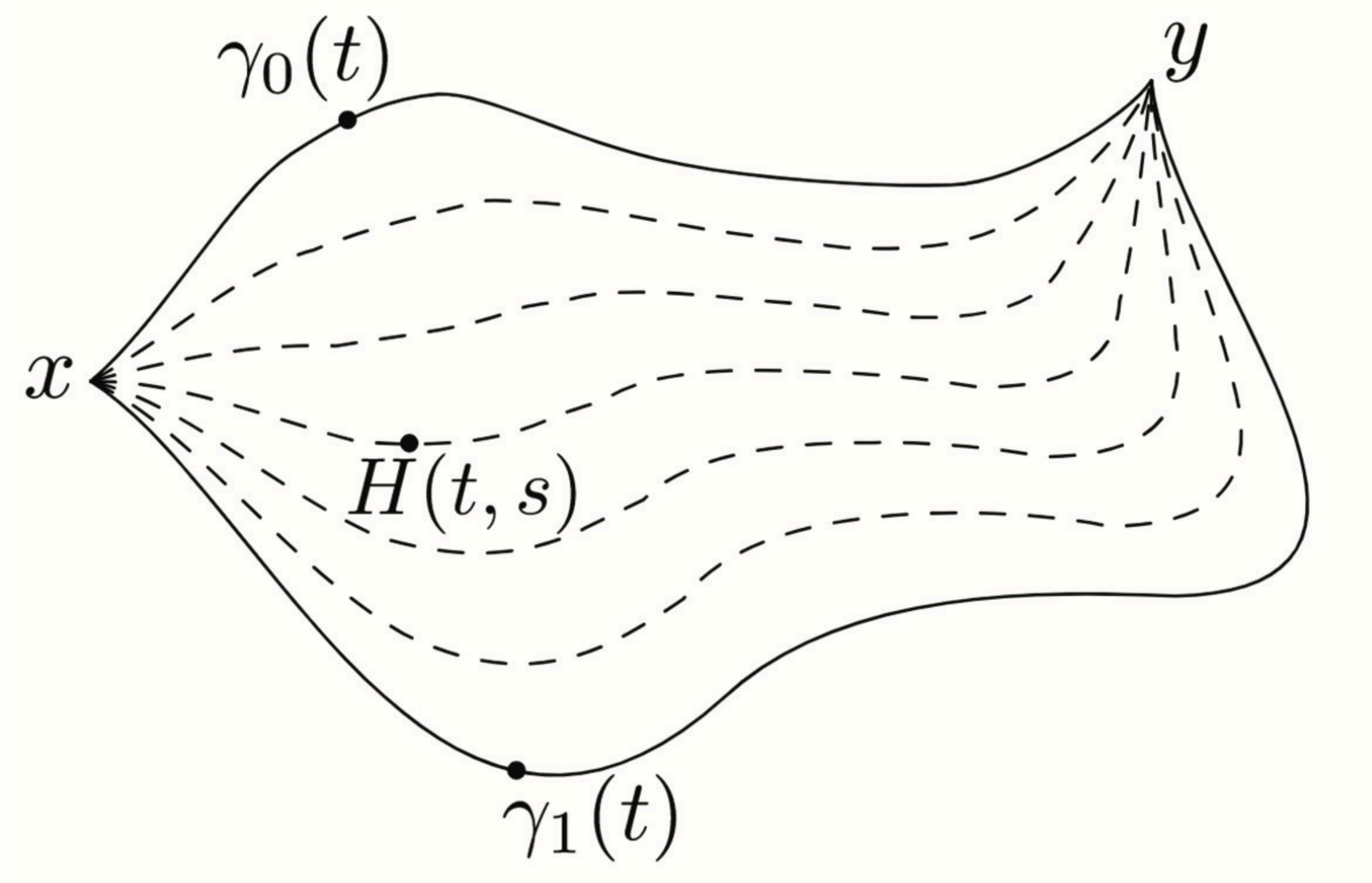
Types	Logic	Sets	Homotopy
A	proposition	set	space
$a : A$	proof	element	point
$B(x)$	predicate	family of sets	fibration
$b(x) : B(x)$	conditional proof	family of elements	section
$\mathbf{0}, \mathbf{1}$	$\bot, \top$	$\emptyset, \{\emptyset\}$	$\emptyset, *$
$A + B$	$A \vee B$	disjoint union	coproduct
$A \times B$	$A \wedge B$	set of pairs	product space
$A \to B$	$A \Rightarrow B$	set of functions	function space
$\sum_{(x:A)} B(x)$	$\exists_{x:A} B(x)$	disjoint sum	total space
$\prod_{(x:A)} B(x)$	$\forall_{x:A} B(x)$	product	space of sections
Id_A	equality $=$	$\{ (x, x) \mid x \in A \}$	path space A^I

A further correspondence between types and spaces is possible via algebraic topology. This has motivated a global research program known as Univalent Foundations.

Homotopy Type Theory: Univalent Foundations of Mathematics, (2013). Institute for Advanced Study. p. 11



A homotopy H exhibited between two continuous functions (γ_0, γ_1), mapping topological spaces, x and y. The equivalence class of spaces homotopic to x is called its homotopy type.

Structuralism, Invariance, Univalence

$$(A \simeq B) \simeq (A = B)$$

The univalence axiom states that structural equivalence (up to homotopy) is equivalent to identity. Homotopy thus serves as an expression of invariance that furnishes our worldview with an identity function. This is a form of structuralism, Awodey stating that “mathematical objects simply are structures” from this viewpoint.

Moreover, computational explanations are intrinsically geometric in this scheme.

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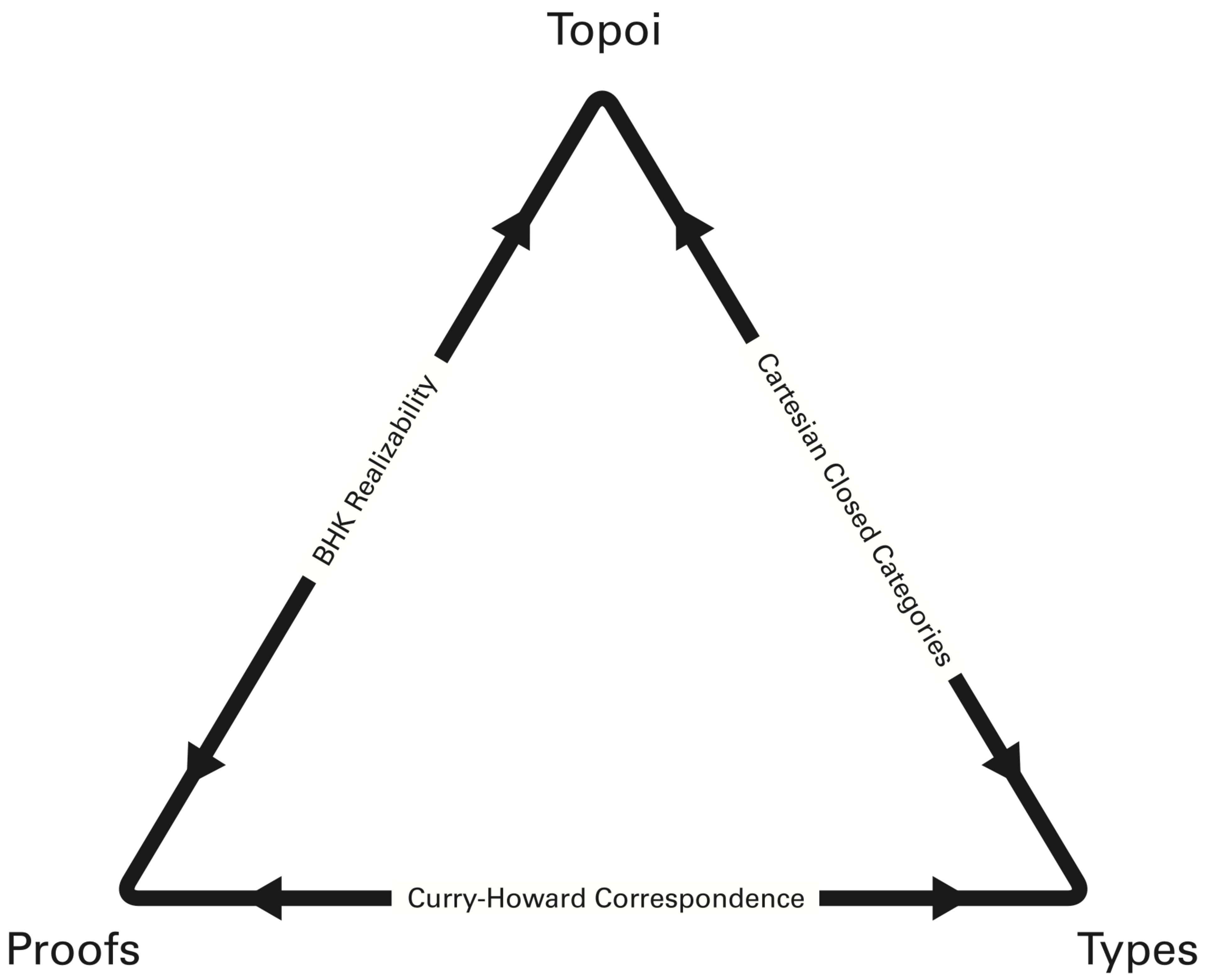

An illustration of the correspondences between logic (proof theory), mathematics (category theory), and language (type theory), in Computational Trinitarianism (Harper). Univalent foundations add a correspondence between topology and type theory via the Univalence Axiom.

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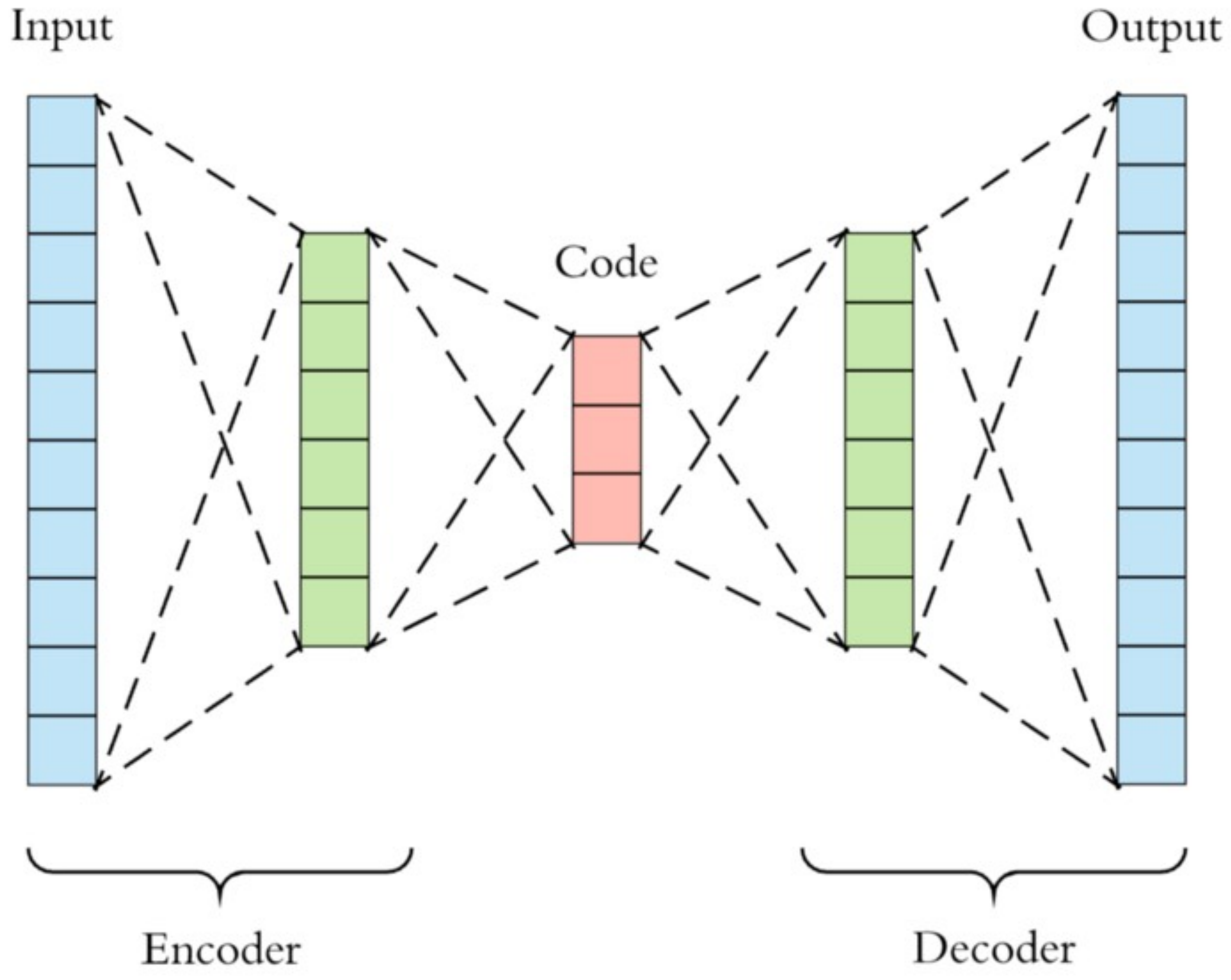

The latent space of a deep learning model refers to the use of dimensionality reduction to learn an ‘embedding’ of a set of concepts. The canonical interpretation is that of a Hilbert space, whereby each embedding space is uniquely induced in response to a given learning task (e.g. learning the ‘tone’ of a text).

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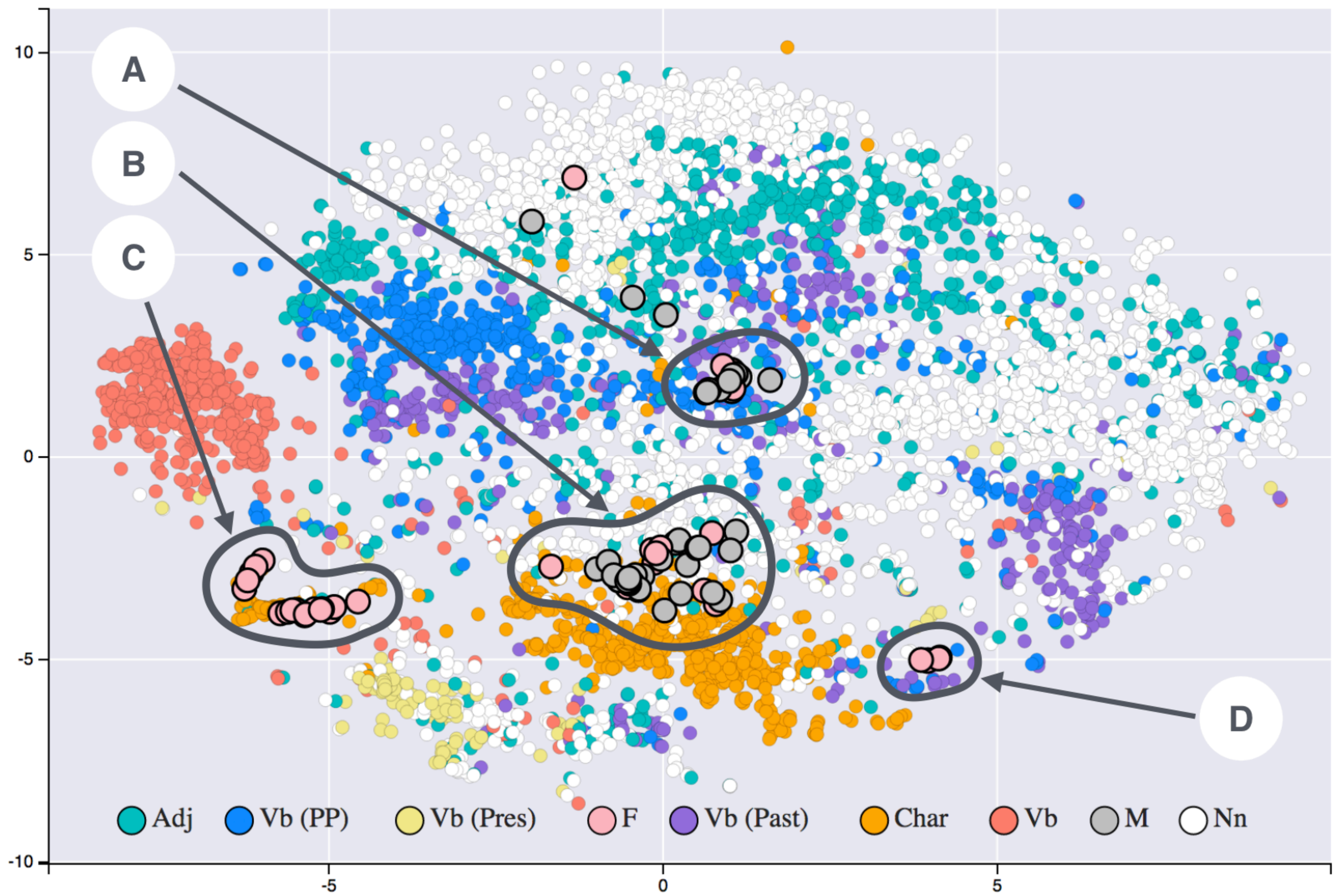

Exploring the Role of Gender in 19th Century Fiction Through the Lens of Word Embeddings, Grayson et al (2017)

“The manifold hypothesis is that natural data forms lower-dimensional manifolds in its embedding space. There are both theoretical and experimental reasons to believe this to be true. If you believe this, then the task of a classification algorithm is fundamentally to separate a bunch of tangled manifolds.”

- Chris Olah, 2014. Neural Networks, Manifolds, and Topology.

The Topological Turn

The worldview we arrive at integrates realizability semantics with the fundamental geometricity of computation. It is the viewpoint itself to which I suggest we attach the name computation. It is computation that defines the specific ways in which these forms—by turns logical, mathematical, and linguistic—hang together.

The topological view gives us a way of synthesizing computational reason with the Sellarsian space of reasons (world models). Moreover it is a candidate for unifying symbolic and model-blind approaches to AI. The image of computation which emerges is that of a toolkit for navigating continuous spaces, bounded only by the expressive limits of language for describing structures (topologies).

The ‘realizability’ of a world then refers precisely to our ability to construct a topological site for the identification of those entities that are said to inhabit that world. The operations that define the logic of that specific site are computations which are bound to the locality of truth procedures.